

Section 7.2 Trigonometric Integrals

We use trig. identities to integrate certain combinations of trig. functions

Examples ① $\int \cos^3 x dx$ we use $\cos^2 x = 1 - \sin^2 x$. Then

$$\int \cos^3 x dx = \int \cos x \cdot \cos^2 x dx = \int \cos x \cdot (1 - \sin^2 x) dx = \int \cos x dx - \int \cos x \sin^2 x dx$$

let $u = \sin x$, $du = \cos x dx$. Then

$$\int \cos^3 x dx = \sin x - \int u^2 du = \sin x - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C$$

$$② \int \sin^3 x dx = \int \sin x \cdot \sin^2 x dx = \int \sin x (1 - \cos^2 x) dx = \int \sin x dx - \int \sin x \cos^2 x dx$$

let $u = \cos x$, $du = -\sin x dx$. Then

$$\int \sin^3 x dx = -\cos x + \int u^2 du = -\cos x + \frac{u^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C.$$

$$③ \int \cos^5 x \sin^2 x dx = \int \cos x (1 - \sin^2 x)^2 \sin^2 x dx. \text{ let } u = \sin x, du = \cos x dx. \text{ Then}$$

$$\begin{aligned} \int (1-u^2)^2 u^2 du &= \int (-2u^2 + u^4) u^2 du = \int u^2 - 2u^4 + u^6 du = \frac{u^3}{3} - \frac{2}{5} u^5 + \frac{u^7}{7} + C \\ &= \frac{\sin^3 x}{3} - \frac{2 \sin^5 x}{5} + \frac{\sin^7 x}{7} + C \end{aligned}$$

$$\begin{aligned} ④ \int \cos^4 x dx &= \int (\cos^2 x)^2 dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4} \int 1 + 2\cos 2x + \cos^2 2x dx \\ &= \frac{1}{4} \int 1 + 2\cos 2x + \frac{1}{2} (1 + \cos 4x) dx = \frac{1}{4} \left(x + \sin 2x + \frac{1}{2} x + \frac{1}{2} \cdot \frac{1}{4} \sin 4x\right) + C \end{aligned}$$

In general, we consider $\int \sin^m(x) \cos^n(x) dx$.

a) if the power of cosine is odd ($n=2k+1$), Then

$$\int \sin^m(x) \cos^{2k+1}(x) dx = \int \sin^m(x) \cos^{2k}(x) \cdot \cos(x) dx = \int \sin^m(x) (1 - \sin^2(x))^k \cdot \cos(x) dx.$$

Then do a u-sub with $u = \sin x$

b) if the power of sine is odd ($m=2k+1$), then

$$\int \sin^{2k+1}(\cos^n(x)) dx = \int \sin^{2k}(\cos(x)) \cos(x) dx = \int (1 - \cos^2(x))^k \cos^n(x) \sin(x) dx$$

Then do a u-sub with $u = \cos x$.

Note: if both powers n and m are odd, we can use either (a) or (b)

(c) if the powers of both sine and cosine are even, we use the half-angle formulas: $\sin^2 x = \frac{1 - \cos 2x}{2}$, $\cos^2 x = \frac{1 + \cos 2x}{2}$; ($\sin 2x = 2 \sin x \cos x$ is useful!)

Let's use a similar procedure for integrals involving $\tan(x)$ and $\sec(x)$.

Examples ① $\int \tan^7 \theta \sec^5 \theta d\theta = \int \tan^6 \theta \cdot \sec^4 \theta \cdot \sec \theta \tan \theta d\theta$

$$= \int (\sec^2 \theta - 1)^3 \cdot \sec^4 \theta \cdot \sec \theta \tan \theta d\theta. \text{ let } u = \sec \theta, du = \sec \theta \tan \theta d\theta$$
$$= \int (u^2 - 1)^3 \cdot u^4 \cdot du = \int (u^6 - 3u^4 + 3u^2 - 1)u^4 du = \int u^{10} - 3u^8 + 3u^6 - u^4 du$$
$$= \frac{u^{11}}{11} - \frac{3u^9}{9} + \frac{3u^7}{7} - \frac{u^5}{5} + C = \frac{\sec^6 \theta}{11} - \frac{3\sec^9 \theta}{9} + \frac{3\sec^7 \theta}{7} - \frac{\sec^5 \theta}{5} + C$$

② $\int \tan^5 \theta \sec^4 \theta d\theta = \int \tan^5 \theta \cdot \sec^2 \theta \cdot \sec^2 \theta d\theta = \int \tan^5 \theta (1 + \tan^2 \theta) \cdot \sec^2 \theta d\theta.$
(let $u = \tan \theta$, $du = \sec^2 \theta d\theta$) $= \int u^5 (1 + u^2) du = \int u^5 + u^7 du$
$$= \frac{u^6}{6} + \frac{u^8}{8} + C = \frac{\tan^6 \theta}{6} + \frac{\tan^8 \theta}{8} + C.$$

In general, we consider $\int \tan^m(x) \sec^n(x) dx$

(a) if the power of sec is even ($n = 2k$)

$$\int \tan^m(x) \sec^{2k}(x) dx = \int \tan^m(x) (\sec^2(x))^{k-1} \cdot \sec^2(x) dx = \int \tan^m(x) (1 + \tan^2(x))^{k-1} \sec^2(x) dx$$

then do a u-sub with $u = \tan(x)$

(b) if the power of tan is odd ($m = 2k+1$)

$$\int \tan^{2k+1}(x) \sec^n(x) dx = \int (\tan^2(x))^k \sec^{n-1}(x) \cdot \sec x \cdot \tan x dx = \int (\sec^2 x - 1)^k \sec^{n-1}(x) \sec x \tan x dx$$

then do a u-sub with $u = \sec(x)$.

(c) Other cases have no general procedures; we may use identities, or integration by parts, etc.

Recall two important anti-derivatives:

$$\int \tan x \, dx = \ln |\sec x| + C \quad \text{and} \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C.$$

Example ① $\int \tan^3 x \, dx = \int \tan x \tan^2(x) \, dx = \int \tan x (\sec^2 x - 1) \, dx$
 $= \int \tan x \cdot \sec^2 x \, dx - \int \tan x \, dx \quad (\text{let } u = \sec x, du = \sec x \tan x \, dx)$
 $= \int u \cdot du - \ln |\sec x| + C = \frac{u^2}{2} - \ln |\sec x| + C = \frac{\sec^2 x}{2} - \ln |\sec x| + C.$
 (we can also let $u = \tan x, du = \sec^2 x \, dx$).

② $\int \sec^3(x) \, dx = \int \sec x \cdot \sec^2 x \, dx \rightarrow \text{Integration by parts.}$

let $u = \sec x, dv = \sec^2 x \, dx$; then $du = \sec x \tan x \, dx, v = \tan x$. Then
 $\int \sec^3(x) \, dx = \sec x \tan x - \int \sec x \cdot \tan^2 x \, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$. Then
 $2 \int \sec^3(x) \, dx = \sec x \tan x + \int \sec x \, dx = \sec x \tan x + \ln |\sec x + \tan x| + C$. Finally
 $\int \sec^3(x) \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$.

How to evaluate integrals of the form

- ⓐ $\int \sin(mx) \cos(nx) \, dx$: use $\sin A \cos B = \frac{1}{2} (\sin(A-B) + \sin(A+B))$,
- ⓑ $\int \sin(mx) \sin(nx) \, dx$: use $\sin A \sin B = \frac{1}{2} (\cos(A-B) - \cos(A+B))$,
- ⓒ $\int \cos(mx) \cos(nx) \, dx$: use $\cos A \cos B = \frac{1}{2} (\cos(A-B) + \cos(A+B))$.

Examples ① $\int \sin(4x) \sin(2x) \, dx = \int \frac{1}{2} [\cos(4x-2x) - \cos(4x+2x)] \, dx$

$$= \frac{1}{2} \int \cos(2x) - \cos(6x) \, dx = \frac{1}{2} \left[\frac{1}{2} \sin 2x - \frac{1}{6} \sin 6x \right] + C$$

② $\int \sin(5x) \cos(7x) \, dx = \frac{1}{2} \int \sin(5x-7x) + \sin(5x+7x) \, dx$

$$= \frac{1}{2} \int \sin(-2x) + \sin(12x) \, dx = \frac{1}{2} \left[-\frac{1}{2} \cos(-2x) - \frac{1}{12} \cos(12x) \right] + C$$